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# Separation in a slow linear shear flow past a cylinder and a plane

By A. M. J. DAVIS AND M. E. O'NEILL
Department of Mathematics, University College London

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In a slow linear shear flow over a cylinder in contact with a plane, there is an infinite set of eddies within the cusps at the point of contact. If the cylinder is not in contact with the plane, there is a flux of fluid between the cylinder and the plane, no matter how small the gap. When the gap is approximately 0·685 times the cylinder radius or less, the flow separates from the boundaries. Single eddies form alternately on the plane and the cylinder. These interlace as the cylinder approaches the plane and force the fluid which flows through the gap to take a tortuous path. The expressions for the force and torque acting on the cylinder are also given.

## 1. Introduction

One of the most interesting phenomena in two-dimensional Stokes flows is the existence of eddies within a corner formed by two intersecting planes, provided that the angle of the corner is sufficiently small. This problem was first studied by Dean & Montagnon (1949), who, by considering solutions of the plane biharmonic equation in polar co-ordinates, showed that the power of the radial co-ordinate in the solution is complex if the angle between the intersecting planes is less than about 146·3°. This result was later interpreted by Moffatt (1964) as indicating the existence of an infinite sequence of line vortices, whose strength diminishes exponentially as the corner is approached. Schubert (1967) showed that eddies exist in the cusps when there is a linear shear flow over a circular cylinder in contact with a plane, and Wakiya (1975) has shown that for a similar flow over a cylindrical projection on a plane, the Moffat eddies exist within the asymptotic structure of the flow near the line of intersection between the projection and the plane provided that the angle of intersection is less than 146·3°.

Schubert's work, although providing proof of the existence of eddies within the cusps, did not give details of the structure and extent of the eddy region, which is of course finite when a cylinder in contact with a plane is in a linear shear flow. In this paper, we present a complete solution to this problem and determine both the extent of the eddy region and the shape of the streamlines. We also consider the corresponding problem when the cylinder is not in contact with the plane. If the cylinder is a great distance from the plane, the shear flow is approximately uniform in the neighbourhood of the cylinder. Consequently the streamlines of the flow about the cylinder have fore–aft symmetry locally and there is no separation of the flow from the cylinder apart from along the streamlines which lie along the diameter of the cylinder parallel to the plane. This however poses the questions as to how the eddies, which are known to exist in the flow when the cylinder is in contact with the plane, can be formed as the cylinder is brought closer to the plane, or indeed whether the eddies exist in the flow only when the cylinder is in contact with the plane.

In this paper we show that there is always a flux of fluid through the gap between the cylinder and plane, whatever the gap size. There is therefore a streamline attached to the cylinder which divides the fluid which passes through the gap from that which flows over and past the cylinder. Apart from along this streamline, there is no separation of the flow from either the cylinder or the plane if the distance $d$ from the centre to the plane is greater than approximately $1{\cdot}685a$, where $a$ is the radius of the cylinder. At this critical distance, the flow starts to separate on the plane at points $2{\cdot}296a$ from the closest point of the plane to the cylinder. An eddy then forms adjacent to the plane and separation from the cylinder begins when its centre is about $1{\cdot}030a$ from the plane. As $d/a$ is further decreased, the primary eddies grow and secondary, tertiary, etc. eddies are formed in separated flow regions on the cylinder and plane alternately. In the neighbourhood of where the gap is narrowest, the flow is approximately a plane Poiseuille flow, so that fluid which flows through the gap must 'snake' its way between the vortices. In this way, the infinite set of eddies which exists when the cylinder is in contact with the plane is produced in a systematic manner. We conclude our analysis with the calculation of the force and torque acting on the cylinder.

## 2. Statement of the problem

The motion sufficiently near a plane boundary of a viscous incompressible fluid, of constant density $\rho$ and viscosity $\mu$, can be conveniently approximated by a uniform linear shear flow. We consider how this flow is affected by the presence of a rigid circular cylinder of radius $a$ with its axis parallel to and at a distance $d \geqslant a$ from the plane. Choosing the origin of Cartesian co-ordinates in the plane boundary with the $x$ axis in the direction of the shear flow and the $y$ axis normal to the plane of the shear and intersecting the axis of the cylinder, the undisturbed shear flow has Cartesian velocity components $(Uy, 0, 0)$, where $U$ is the constant rate of shear. Since the fluid is incompressible, the equation of continuity is

$$\operatorname{div} \mathbf{v} = 0, \tag{2.1}$$

where $\mathbf{v}$ is the fluid velocity. A stream function $\psi$ can be defined such that the components of $\mathbf{v}$ are given by

$$\mathbf{v} = (U\, \partial\psi/\partial y,\ -U\, \partial\psi/\partial x,\ 0). \tag{2.2}$$

The boundary conditions are that

$$\psi = \partial\psi/\partial y = 0 \tag{2.3}$$

on the plane, while on the cylinder,

$$\psi = M, \quad \partial\psi/\partial n = 0, \tag{2.4}$$

with $\partial/\partial n$ denoting the derivative along the outward normal to the cylinder. The constant $UM$ can be identified as the flux of fluid through the gap between the cylinder and the plane. When the cylinder is in contact with the plane, $M = 0$. Otherwise it is an unknown of the problem. The boundary condition at infinity is

$$\psi \sim \tfrac{1}{2}y^2 \quad (y \to \infty). \tag{2.5}$$

| $n$ | $\lambda_n$ | $\mu_n$ |
|---|---|---|
| 1 | $4{\cdot}21239+2.25073i$ | $7.49768+2.76858i$ |
| 2 | $10{\cdot}7125\ +3{\cdot}10319i$ | $13{\cdot}9000\ +3{\cdot}35221i$ |
| 3 | $17{\cdot}0734\ +3{\cdot}55108i$ | $20{\cdot}2385\ +3{\cdot}71677i$ |

TABLE 1

We assume that in the fluid motion the inertia terms in the Navier–Stokes equations can be neglected. The equations of motion are accordingly

$$\nabla p = \mu \nabla^2 \mathbf{v}, \tag{2.6}$$

where $p$ denotes the hydrodynamic fluid pressure. On eliminating $p$ from (2.6), we obtain the plane biharmonic equation for $\psi$:

$$\nabla^4 \psi = 0. \tag{2.7}$$

Although the assumption of Stokes flow is eventually violated as $y \to \infty$, this is unimportant since the region of interest lies in the vicinity of the cylinder, which is assumed to be near to the plane. In this region, the appropriate Reynolds number for the flow is $Ua^2\rho/\mu$ and this must be small for (2.6) to hold. We note however that the undisturbed shear flow is a solution of both the Stokes equations and the full Navier–Stokes equations.

## 3. Cylinder in contact with the plane

In subsequent work we shall assume that all variables have been non-dimensionalized using $U$, $\rho$, $\mu$ and the diameter $2a$ of the cylinder as reference scales. Defining inverse co-ordinates $(u, v)$ by means of the relations

$$x = v/(u^2+v^2), \quad y = u/(u^2+v^2), \tag{3.1}$$

the fluid region is given by $0 < u < 1$, $-\infty < v < \infty$. The cylinder is $u = 1$ and the plane is $u = 0$. On writing

$$\psi = \tfrac{1}{2}y^2 - \chi$$

it is seen that $\chi$ must satisfy (2.7), together with the boundary conditions

$$\chi = \partial\chi/\partial u = 0 \quad (u = 0),$$

$$\chi = \tfrac{1}{2}y^2, \quad \partial\chi/\partial u = y\,\partial y/\partial u \quad (u = 1),$$

$$\chi = o(y^2) \quad (u^2+v^2 \to 0).$$

The solution for $\chi$, which is equivalent to that given by Schubert (1967), is

$$\chi = \tfrac{1}{2}(u^2+v^2)^{-1}\int_0^\infty \frac{[s(\sinh su - su\cosh su) + (e^{-s}\sinh s - s + s^2)\,u\sinh su]}{\sinh^2 s - s^2}\cos sv\, ds. \tag{3.2}$$

The integrand of (3.2) is regular at $s = 0$ and has simple poles in the fourth quadrant at $\{-i\lambda_n, -i\mu_n;\ n \geqslant 1\}$, where $\{\lambda_n\}$ and $\{\mu_n\}$ are respectively the zeros of $\sin z \pm z = 0$ in the first quadrant, arranged in order of increasing real part. The first few values, tabulated by Buchwald (1964), are listed in table 1.

| $m$ | $v$ | $x$ | $y$ |
|---|---|---|---|
| 1 | 1·0610 | 0·7712 | 0·3634 |
| 2 | 2·4568 | 0·3908 | 0·0796 |
| 3 | 3·8526 | 0·2553 | 0·0331 |

TABLE 2

The theory of residues then shows that for $v > 0$

$$\psi = -\tfrac{1}{2}\pi(u^2+v^2)^{-1}\,\mathrm{Re}\sum_{n=1}^{\infty}\left\{\frac{(1-u)\sin\lambda_n u + u\sin\lambda_n(1-u)}{1+\cos\lambda_n}\exp(-\lambda_n v) - \frac{(1-u)\sin\mu_n u - u\sin\mu_n(1-u)}{1-\cos\mu_n}\exp(-\mu_n v)\right\}. \quad (3.3)$$

We note that the $\lambda_n$ and $\mu_n$ terms are respectively even and odd functions of $u-\frac{1}{2}$ and that they are oscillatory functions of $v$ with exponentially small amplitudes when $v > 1$, with a decay factor $\sim e^{-2\pi v}$ in each series. In particular, when $u = \frac{1}{2}$,

$$\psi(\tfrac{1}{2}, v) \sim \frac{-\frac{1}{2}\pi}{\frac{1}{4}+v^2}\,\mathrm{Re}\,\frac{\sin\frac{1}{2}\lambda_1}{1+\cos\lambda_1}\exp(-\lambda_1 v) = -\frac{\pi}{\frac{1}{4}+v^2}\,\mathrm{Re}\,\frac{\sin^3\frac{1}{2}\lambda_1}{\lambda_1^2}\exp(-\lambda_1 v),$$

with relative error of order $e^{-2\pi v}$. The right-hand side vanishes at infinitely many values of $v$ given by

$$v\,\mathrm{Im}\,\lambda_1 = (m-\tfrac{1}{2})\pi + 3\arg(\sin\tfrac{1}{2}\lambda_1) - 2\arg\lambda_1. \quad (3.4)$$

Similar equations hold for $u \neq \frac{1}{2}$ but with relative error of order $e^{-\pi v}$. Since $\psi$ vanishes on $u = 0$ and $u = 1$, the existence of infinitely many curves $\psi = 0$ in the fluid linking the cylinder and plane is established. These streamlines form the dividing streamlines of an infinite sequence of eddies in the neighbourhood of the point of contact between the cylinder and plane, where $v = \infty$. In table 2 we have listed the first three of the solutions of (3.4) together with the corresponding values of $x$ and $y$ which are determined from (3.1).

The points at which the curves $\psi = 0$ separate from the cylinder and plane are given by the zeros of $\partial^2\psi/\partial u^2 = 0$ at $u = 1$ and $u = 0$ respectively. Retaining only the leading terms in the series of (3.3), we have

$$\frac{\partial^2\psi}{\partial u^2}(1, v) \sim \frac{\pi}{1+v^2}\,\mathrm{Re}\,\{\lambda_1\exp(-\lambda_1 v) - \mu_1\exp(-\mu_1 v)\},$$

$$\frac{\partial^2\psi}{\partial u^2}(0, v) \sim \frac{\pi}{v^2}\,\mathrm{Re}\,\{\lambda_1\exp(-\lambda_1 v) + \mu_1\exp(-\mu_1 v)\}.$$

Hence the values of $v$ for separation are given asymptotically by

$$\mathrm{Re}\,\{\lambda_1\exp(-\lambda_1 v)\,[1 \mp (\mu_1/\lambda_1)\exp(-(\mu_1-\lambda_1)v)]\} = 0. \quad (3.5)$$

The minus and plus signs apply respectively to the cylinder and the plane. The $\mu_1$ terms are important at the first separation points, barely significant at the second and negligible thereafter. In table 3, the solutions to (3.5) for the four smallest values of $v$ are displayed together with the values of $x$ and $y$ at the separation points.

| Cylinder ($u = 1$) | | | Plane ($u = 0$) | |
|---|---|---|---|---|
| $v$ | $x$ | $y$ | $v$ | $x$ |
| 0·937 | 0·499 | 0·532 | 0·895 | 1·117 |
| 2·312 | 0·364 | 0·158 | 2·311 | 0·433 |
| 3·708 | 0·251 | 0·068 | 3·708 | 0·270 |
| 5·104 | 0·189 | 0·037 | 5·104 | 0·196 |

TABLE 3

The angles at which the separation streamlines detach from the boundaries can be found on evaluating the third derivatives of $\psi$. From (3.3) we have

$$\frac{\partial^3\psi}{\partial u^2\,\partial v}(1,v) \sim \frac{\pi}{1+v^2}\mathrm{Re}\{-\lambda_1^2\exp(-\lambda_1 v)+\mu_1^2\exp(-\mu_1 v)\},$$

$$\frac{\partial^3\psi}{\partial u^2\,\partial v}(0,v) \sim \frac{\pi}{v^2}\mathrm{Re}\{-\lambda_1^2\exp(-\lambda_1 v)-\mu_1^2\exp(-\mu_1 v)\},$$

whilst, when $v$ satisfies (3.5),

$$\frac{\partial^3\psi}{\partial u^3}(1,v) \sim \frac{\pi}{1+v^2}\mathrm{Re}\{-\lambda_1\exp(-\lambda_1 v)\cos\lambda_1-\mu_1\exp(-\mu_1 v)\cos\mu_1\},$$

$$\frac{\partial^3\psi}{\partial u^3}(0,v) \sim \frac{\pi}{v^2}\mathrm{Re}\{\lambda_1\exp(-\lambda_1 v)\cos\lambda_1-\mu_1\exp(-\mu_1 v)\cos\mu_1\}.$$

The angles of separation are given by

$$\tan^{-1}\left[3\frac{\partial^3\psi}{\partial u^2\,\partial v}\Big/\frac{\partial^3\psi}{\partial u^3}\right]_{(1,\,v)},\quad \tan^{-1}\left[-3\frac{\partial^3\psi}{\partial u^2\,\partial v}\Big/\frac{\partial^3\psi}{\partial u^3}\right]_{(0,\,v)},$$

where $v$ satisfies (3.5), and are readily shown to be

$$\tan^{-1}\left\{3\,\mathrm{Im}\left[\lambda_1\mp\frac{(\mu_1-\lambda_1)(\mu_1/\lambda_1)\exp[-(\mu_1-\lambda_1)v]}{1\mp(\mu_1/\lambda_1)\exp[-(\mu_1-\lambda_1)v]}\right]\Big/ \mathrm{Im}\left[\cos\lambda_1\pm\frac{(\cos\lambda_1+\cos\mu_1)(\mu_1/\lambda_1)\exp[-(\mu_1-\lambda_1)v]}{1\mp(\mu_1/\lambda_1)\exp[-(\mu_1-\lambda_1)v]}\right]\right\}, \tag{3.6}$$

with the upper and lower signs applying on the cylinder and plane respectively. The limiting value of this angle as $v\to\infty$ is

$$\tan^{-1}\left\{\frac{3\,\mathrm{Im}(\lambda_1)}{\mathrm{Im}(\cos\lambda_1)}\right\} = 58{\cdot}61^\circ,$$

while the angles subtended by the first separation line are 61·5° and 55·8° at the cylinder and plane respectively. In figure 1, we have sketeched the shapes of the streamlines which divide the two largest eddies. It will be observed that the outermost dividing streamline separates from the plane at a point just over twice the radius from the point of contact. This streamline separates from the cylinder at a point slightly above the extremity of the diameter parallel to the plane. The outermost dividing streamline

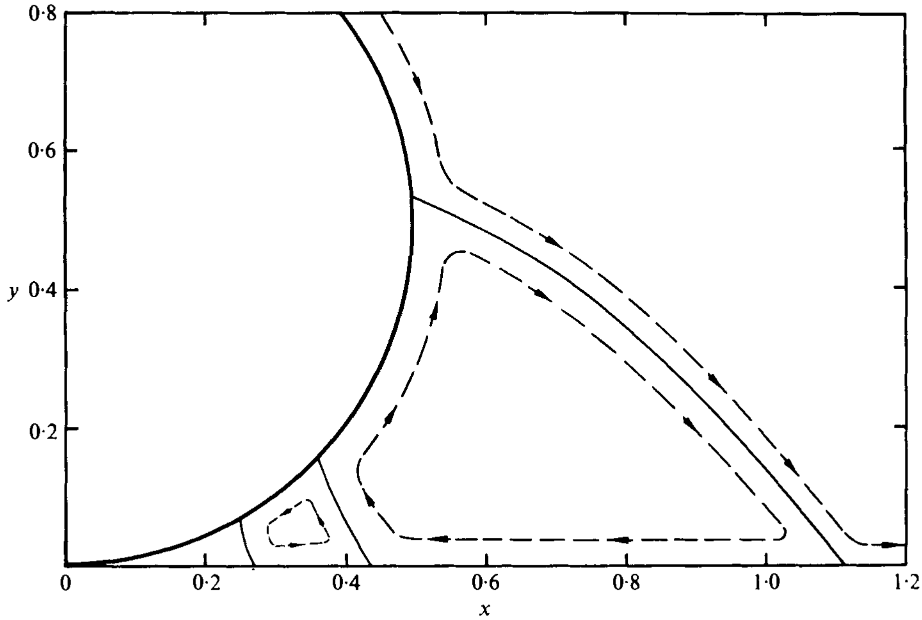


FIGURE 1. The separation streamlines when the cylinder touches the plane.

is convex outwards while all of the other dividing streamlines are concave outwards. The streamline pattern is symmetrical about the plane $x = 0$. The dashed curves indicate the general direction of the flow within and outside the eddy region.

## 4. Cylinder not in contact with the plane

For this case, the solution for $\psi$ is most conveniently found in terms of bipolar co-ordinates $(\xi, \eta)$ which are related to the Cartesian co-ordinates by

$$x = \frac{c \sin \eta}{\cosh \xi - \cos \eta}, \quad y = \frac{c \sinh \xi}{\cosh \xi - \cos \eta} \tag{4.1}$$

with $c = \frac{1}{2} \sinh \alpha$. The plane is given by $\xi = 0$ while the cylinder is given by $\xi = \alpha$. The distance from the centre of the cylinder to the plane is $\frac{1}{2} \cosh \alpha$. The flow region is given by $0 < \xi < \alpha$, $|\eta| \leqslant \pi$.

When the cylinder is not in contact with the plane, the value of the stream function on the cylinder is a constant $M$ which depends on the distance of the cylinder from the plane. We therefore write the stream function as

$$\psi = \tfrac{1}{2}y^2 - \chi + M\phi, \tag{4.2}$$

where $\chi$ and $\phi$ satisfy (2.7} and the boundary conditions

$$\chi = \partial\chi/\partial\xi = \phi = \partial\phi/\partial\xi = 0 \quad (\xi = 0), \tag{4.3}$$

$$\chi = \tfrac{1}{2}y^2, \quad \frac{\partial \chi}{\partial \xi} = y\frac{\partial y}{\partial \xi}, \quad \phi = 1, \quad \frac{\partial \phi}{\partial \xi} = 0 \quad (\xi = \alpha), \tag{4.4}$$

$$\chi, \phi = o(y^2) \quad \text{as} \quad \xi^2 + \eta^2 \to 0. \tag{4.5}$$

We must look for solutions $\chi$ and $\phi$ such that $\psi$ is an even periodic function of $\eta$ with period $2\pi$, and noting that $y\Phi$ and $(x^2+y^2)\,\Phi$ are biharmonic functions if $\Phi$ is harmonic, it follows that suitable biharmonic functions which satisfy the boundary conditions (4.3) are given by

$$\chi = \tfrac{1}{2}c^2(\cosh\xi - \cos\eta)^{-1} \sum_{n=0}^{\infty} \chi_n(\xi)\cos n\eta \tag{4.6}$$

and

$$\phi = (\cosh\xi - \cos\eta)^{-1}[\phi_0(\xi) + \phi_1(\xi)\cos\eta], \tag{4.7}$$

where

$$\left.\begin{aligned}
\chi_0(\xi) &= A_0\xi\sinh\xi + B_0(\xi\cosh\xi - \sinh\xi),\\
\chi_1(\xi) &= A_1(\cosh 2\xi - 1) + B_1(\sinh 2\xi - 2\xi),\\
\chi_n(\xi) &= A_n[\cosh(n+1)\xi - \cosh(n-1)\xi] + B_n[(n-1)\sinh(n+1)\xi\\
&\quad - (n+1)\sinh(n-1)\xi] \quad (n \geqslant 2),\\
\phi_0(\xi) &= a_0\xi\sinh\xi + b_0(\xi\cosh\xi - \sinh\xi),\\
\phi_1(\xi) &= a_1(\cosh 2\xi - 1) + b_1(\sinh 2\xi - 2\xi),
\end{aligned}\right\} \tag{4.8}$$

the coefficients $a_0, b_0, \ldots, A_n$ and $B_n$ $(n \geqslant 2)$ being independent of $\xi$ and $\eta$. The boundary conditions (4.4) imply that

$$\phi_0(\alpha) = \cosh\alpha, \quad \phi_1(\alpha) = -1, \quad \phi_0'(\alpha) = \sinh\alpha, \quad \phi_1'(\alpha) = 0,$$

$$\chi_0(\alpha) = \sinh\alpha, \quad \chi_n(\alpha) = e^{-(n-1)\alpha} - e^{-(n+1)\alpha} \quad (n \geqslant 1),$$

$$\chi_0'(\alpha) = \cosh\alpha, \quad \chi_n'(\alpha) = (n+1)\,e^{-(n+1)\alpha} - (n-1)\,e^{-(n-1)\alpha} \quad (n \geqslant 1).$$

It readily follows that

$$a_0 = -\frac{\sinh^2\alpha}{\alpha^2 - \sinh^2\alpha}, \quad b_0 = \frac{\alpha + \sinh\alpha\cosh\alpha}{\alpha^2 - \sinh^2\alpha},$$

$$a_1 = -\frac{\tfrac{1}{2}\tanh\alpha}{\alpha - \tanh\alpha}, \quad b_1 = \frac{\tfrac{1}{2}}{\alpha - \tanh\alpha},$$

$$A_0 = \frac{\alpha - \cosh\alpha\sinh\alpha}{\alpha^2 - \sinh^2\alpha}, \quad B_0 = \frac{\sinh^2\alpha}{\alpha^2 - \sinh^2\alpha},$$

$$A_1 = \frac{\alpha\,e^{-2\alpha} - e^{-\alpha}\sinh\alpha + \sinh^2\alpha}{\sinh 2\alpha(\alpha - \tanh\alpha)}, \quad B_1 = \frac{-\tfrac{1}{2}\tanh\alpha}{\alpha - \tanh\alpha},$$

$$A_n = \frac{n(n - \coth\alpha)\sinh^2\alpha + e^{-n\alpha}\sinh n\alpha}{\sinh^2 n\alpha - n^2\sinh^2\alpha}, \quad B_n = \frac{-n\sinh^2\alpha}{\sinh^2 n\alpha - n^2\sinh^2\alpha} \quad (n \geqslant 2).$$

The condition (4.5) is satisfied since $\chi$ and $\phi$ are $\sim \xi^2/(\xi^2+\eta^2) \sim y^2/(x^2+y^2)$ as $x^2+y^2 \to \infty$.

The constant $M$ is determined from the condition that the pressure $p$ is single valued throughout the fluid. The Stokes equations (2.6) imply that $p$ and $\mu\nabla^2\psi$ are conjugate functions. Thus for $p$ to be single valued, we must have

$$\int_C \frac{\partial}{\partial n}(\nabla^2\psi)\,ds = 0, \tag{4.9}$$

where $C$ is any closed curve drawn in the fluid. The simplest choice of $C$ is the $x$ axis and a semicircle at infinity. Now as $r = (x^2+y^2)^{\frac{1}{2}} \to \infty$,

$$\psi = \tfrac{1}{2}y^2[1+o(1)].$$

Hence $$\partial(\nabla^2\psi)/\partial n = o(r^{-1}),$$

so the contribution to the integral from the semicircle at infinity is zero, and since $\psi$ is an even function of $x$, it follows that

$$\int_0^\infty \left[\frac{\partial}{\partial y}(\nabla^2\psi)\right]_{y=0} dx = 0. \tag{4.10}$$

The conformal transformation (4.1) and the decomposition (4.2) means that (4.10) is equivalent to

$$M\int_0^\pi (1-\cos\eta)^2\left(\frac{\partial^3\phi}{\partial\xi^3}\right)_{\xi=0} d\eta = \int_0^\pi (1-\cos\eta)^2\left(\frac{\partial^3\chi}{\partial\xi^3}\right)_{\xi=0} d\eta.$$

On substituting for $\phi$ and $\chi$ from (4.6) and (4.7), we obtain

$$M = c^2\frac{[2\chi_0'''(0)-\chi_1'''(0)]}{[2\phi_0'''(0)-\phi_1'''(0)]} = \frac{(B_0-2B_1)c^2}{(b_0-2b_1)}.$$

On substituting for $c$ and the coefficients, we obtain after simplification

$$M = (2\alpha^2+\alpha\sinh 2\alpha - 4\sinh^2\alpha)/16\alpha. \tag{4.11}$$

Evidently $M > 0$ for all positive values of $\alpha$, and when $\alpha \ll 1$

$$M \sim \frac{\alpha^5}{180}+O(\alpha^7). \tag{4.12}$$

Thus if $\epsilon$ $(= d-a)$ is the minimum clearance between the cylinder and the plane, $\epsilon = \frac{1}{2}(\cosh\alpha - 1) \sim \frac{1}{4}\alpha^2$ and consequently

$$M \sim \frac{4\sqrt{2}}{45}\epsilon^{\frac{5}{2}}+O(\epsilon^{\frac{7}{2}}) \tag{4.13}$$

for small gap widths.

The streamline $\psi = M$ consists of the cylinder and a curve in the fluid with asymptote $y = (2M)^{\frac{1}{2}}$. This curve is the dividing streamline which separates the fluid which flows through the gap from that which flows over and past the cylinder. Equation (4.12) or (4.13) shows that no matter how small the gap, there is always a flux of fluid, albeit very small, through the gap between the cylinder and plane. In the case when $\alpha \gg 1$, we see from (4.11) that $M \sim \frac{1}{32}e^{2\alpha} \sim \frac{1}{2}d^2$.

For separation to occur on the cylinder it is necessary for $\partial^2\psi/\partial\xi^2$ to vanish. The condition for this to happen is found to be

$$\frac{2(\cosh^2\alpha+\sinh^2\alpha)}{\cosh\alpha-\cos\eta} - \frac{5\sinh^2\alpha\cosh\alpha}{(\cosh\alpha-\cos\eta)^2}+\frac{2\sinh^4\alpha}{(\cosh\alpha-\cos\eta)^3}$$
$$+\frac{16M}{\sinh^2\alpha}\left[\frac{\alpha\sinh\alpha}{\alpha^2-\sinh^2\alpha}+\frac{\tanh\alpha\cos\eta}{\alpha-\tanh\alpha}\right]-\sum_{n=0}^{\infty}\chi_n''(\alpha)\cos n\eta = 0. \tag{4.14}$$

The corresponding condition for separation from the plane is

$$\frac{2}{1-\cos\eta}-\frac{16M}{\sinh^2\alpha}\left[\frac{\sinh^2\alpha}{\alpha^2-\sinh^2\alpha}+\frac{\tanh\alpha\cos\eta}{\alpha-\tanh\alpha}\right]-\sum_{n=0}^{\infty}\chi_n''(0)\cos n\eta = 0. \quad (4.15)$$

When $\alpha \gg 1$, (4.15) gives asymptotically $\cos^2\eta = 7$, showing that separation does not occur from the plane when $\alpha$ is sufficiently large. Equation (4.14) gives asymptotically

$$4\alpha^{-1}\cos\eta = 0, \quad (4.16)$$

which has zeros at $\eta = \pm\frac{1}{2}\pi$, i.e. at the extremities of the diameter of the cylinder which is parallel to the plane. Now when $\alpha$ is large, the distance from the cylinder to the plane is large, so that the strength of the shear flow in the neighbourhood of the cylinder is approximately uniform. The solutions of (4.16) just give the usual fore-and-aft stagnation points at which the flow separates as it passes either side of the cylinder.

When $\alpha \gg 1$, the asymptotic form for $\phi$ is

$$\phi \sim (\tfrac{1}{2}\xi^2+2\sin^2\tfrac{1}{2}\eta)^{-1}\frac{\xi^2}{\alpha^2}\left(3-\frac{2\xi}{\alpha}\right)2\sin^2\tfrac{1}{2}\eta.$$

Thus

$$\phi \sim \frac{\xi^2}{\alpha^2}\left(3-\frac{2\xi}{\alpha}\right) \quad (4.17)$$

when $\eta$ is not small. The points on the cylinder which are closest to the plane have $\eta \sim \pi$ and therefore $y \sim \frac{1}{4}\alpha\xi$. Also the minimum clearance $\epsilon \sim \frac{1}{4}\alpha^2$. Thus (4.17) is equivalent to

$$\phi \sim \frac{y^2}{\epsilon^2}\left(3-\frac{2y}{\epsilon}\right),$$

so that when $\alpha \ll 1$ and $\eta \sim \pi$, $\phi$ approximates to the stream function of a plane Poiseuille flow through the gap with unit flux. The total contribution to $\psi$ from $\phi$ is $M\phi$, which is $O(\alpha^5)$ when $\alpha \ll 1$, i.e. algebraically small.

The form of the series solution (4.6) for $\chi$ is unsuitable for examining the behaviour of $\chi$ when $\alpha \ll 1$ since all terms are then significant. To obtain a more suitable form, we define the complex function $F(z)$ of the complex variable $z = x+iy$ as

$$F(z) = \frac{A(z)\sinh z\xi\sinh\xi+B(z)\,[z\cosh z\xi\sinh\xi-\sinh z\xi\cosh\xi]}{\sin\pi z(\sinh^2 z\alpha-z^2\sinh^2\alpha)}\cos(\pi-\eta)z, \quad (4.18)$$

with

$$A(z) = z(z-\coth\alpha)\sinh^2\alpha+e^{-z\alpha}\sinh z\alpha, \quad B(z) = -z\sinh^2\alpha. \quad (4.19)$$

It then follows that the residues of $F(z)$ at the zeros of $\sin\pi z$ are $\pi^{-1}\chi_0(\xi)$ at $z = 0$ and $(2\pi)^{-1}\chi_n(\xi)\cos n\eta$ at $z = n$ for $n \geq 1$. On integrating $F(z)$ around a contour consisting of the imaginary axis indented at the origin together with the infinite semicircle in the half-plane $\mathrm{Re}\, z > 0$, the integral over the semicircle vanishes and

$$\int_{-\infty}^{\infty}F(iy)\,dy = -2\int_0^{\infty}\frac{y\sin y\xi}{\sinh\pi y}\sinh\xi\cosh(\pi-\eta)\,dy = -\frac{\sinh^2\xi}{\cosh\xi-\cos\eta}.$$

Hence from (4.1) and (4.6) we obtain

$$\tfrac{1}{2}y^2-\chi = \frac{\frac{1}{2}c^2}{(\cosh\xi-\cos\eta)}\left[\frac{\sinh^2\xi}{\cosh\xi-\cos\eta}-\sum_{n=0}^{\infty}\chi_n(\xi)\cos n\eta\right]$$

$$= -\frac{c^2\pi}{(\cosh\xi-\cos\eta)}\mathrm{Re}\sum_{n=1}^{\infty}\left\{\frac{\cosh[\sigma_n(\pi-\eta)/\alpha]}{\sinh(\sigma_n\pi/\alpha)}\right.$$

$$\times\frac{[\sinh\xi\sin\sigma_n(1-\xi/\alpha)+\sin(\sigma_n\xi/\alpha)\sinh(\alpha-\xi)]}{\alpha\cos\sigma_n+\sinh\alpha}+\frac{\cosh[\tau_n(\pi-\eta)/\alpha]}{\sinh(\tau_n\pi/\alpha)}$$

$$\left.\times\frac{[\sinh\xi\sin\tau_n(1-\xi/\alpha)-\sin(\tau_n\xi/\alpha)\sinh(\alpha-\xi)]}{\alpha\cos\tau_n-\sinh\alpha}\right\}, \tag{4.20}$$

where $$\alpha\sin\sigma_n+\sigma_n\sinh\alpha = 0, \quad \alpha\sin\tau_n-\tau_n\sinh\alpha = 0, \tag{4.21}$$

the ordering of $\sigma_n$ and $\tau_n$ being such that the real part increases with $n$. We note that the $\sigma_n$ and $\tau_n$ terms in (4.20) are respectively even and odd functions of $\xi-\frac{1}{2}\alpha$. The series representation for $\frac{1}{2}y^2-\chi$ given by (4.20) is most suitable for determining the asymptotic structure of the flow as $\alpha\to 0$ since the terms in each of the series decrease exponentially in absolute magnitude as $\alpha\to 0$ and in this limit $\sigma_n\to\lambda_n$ and $\tau_n\to\mu_n$ with $\lambda_n$ and $\mu_n$ as defined in §3.

On differentiating (4.20) twice with respect to $\xi$, we obtain

$$\left[\frac{\partial^2}{\partial\xi^2}(\tfrac{1}{2}y^2-\chi)\right]_{\xi=\alpha} = \frac{\frac{1}{2}\pi\sinh^2\alpha}{(\cosh\alpha-\cos\eta)}\mathrm{Re}\sum_{n=1}^{\infty}\left\{\frac{\cosh[\sigma_n(\pi-\eta)/\alpha]}{\sinh(\sigma_n\pi/\alpha)}\frac{\sigma_n(\cosh\alpha+\cos\sigma_n)}{\alpha(\alpha\cos\sigma_n+\sinh\alpha)}\right.$$

$$\left.+\frac{\cosh[\tau_n(\pi-\eta)/\alpha]}{\sinh(\tau_n\pi/\alpha)}\frac{\tau_n(\cosh\alpha-\cos\tau_n)}{\alpha(\alpha\cos\tau_n-\sinh\alpha)}\right\}, \tag{4.22}$$

$$\left[\frac{\partial^2}{\partial\xi^2}(\tfrac{1}{2}y^2-\chi)\right]_{\xi=0} = \frac{\frac{1}{2}\pi\sinh^2\alpha}{(1-\cos\eta)}\mathrm{Re}\sum_{n=1}^{\infty}\left\{\frac{\cosh[\sigma_n(\pi-\eta)/\alpha]}{\sinh(\sigma_n\pi/\alpha)}\frac{\sigma_n(\cosh\alpha+\cos\sigma_n)}{\alpha(\alpha\cos\sigma_n+\sinh\alpha)}\right.$$

$$\left.-\frac{\cosh[\tau_n(\pi-\eta)/\alpha]}{\sinh(\tau_n\pi/\alpha)}\frac{\tau_n(\cosh\alpha-\cos\tau_n)}{\alpha(\alpha\cos\tau_n-\sinh\alpha)}\right\}. \tag{4.23}$$

Expression (4.20) shows that $\frac{1}{2}y^2-\chi$ is $O(|\exp(-\sigma_1\eta/\alpha)|)$ when $\eta>\alpha$. When $\alpha\ll 1$, equations (4.1) give

$$x\sim\frac{\alpha\sin\eta}{\xi^2+4\sin^2\frac{1}{2}\eta}, \quad y\sim\frac{\alpha\xi}{\xi^2+4\sin^2\frac{1}{2}\eta}, \tag{4.24}$$

so that for $x$ and $y$ not to be small, $\eta$ must be $O(\alpha)$. Thus in the region $\eta>\alpha$, when $\alpha\ll 1$, it is the term $M\phi$ in the stream function which dominates and this, we have seen, describes approximately a plane Poiseuille flow through the gap.

On writing $\xi=\alpha u$ and $\eta=\alpha v$ in (4.24) and letting $\alpha\to 0$, we obtain (3.1) with $0\leqslant u\leqslant 1$, $v\geqslant 0$. Now $\phi$ is bounded and $M\to 0$ as $\alpha\to 0$, so it is only $\frac{1}{2}y^2-\chi$ which survives in this limit. The expression (3.2) can be recovered from the expression (4.6) for $\chi$ by writing $n\alpha=s$ and letting $\alpha\to 0$, whereupon the summation becomes an integral. Thus

$$c^2(\cosh\xi-\cos\eta)^{-1}\to\tfrac{1}{2}(u^2+v^2)^{-1},$$

$$\frac{1}{2}\sum_{n=0}^{\infty}\chi_n(\xi)\cos n\eta\to\frac{1}{2}\int_0^{\infty}\lim_{\alpha\to 0}\chi_{s/\alpha}(\alpha u)\cos sv\frac{ds}{\alpha}$$

$$=\int_0^{\infty}\frac{[(e^{-s}\sinh s+s^2-s)\,u\sinh su+s(\sinh su-su\cosh su)]}{\sinh^2 s-s^2}\cos sv\,ds$$

and the solution when the cylinder is in contact with the plane is recovered. Since $\sigma_n \to \lambda_n$ and $\tau_n \to \mu_n$ as $\alpha \to 0$, (3.3) is, as expected, the limit of (4.20).

The possibility of separation from the boundaries occurring in the region where $M\phi$ is the dominant part of $\psi$ can be quickly excluded by observing that the vanishing of $\partial^2\phi/\partial\xi^2$ on either $\xi = 0$ or $\xi = \alpha$ implies that

$$\cos\eta = 1 - O(\alpha^2),$$

and therefore $\eta = O(\alpha)$.

From (4.22) and (4.23), we see that if $\alpha \ll 1$ and $\eta = O(\alpha)$,

$$\left[(\cosh\xi - \cos\eta)\frac{\partial^2}{\partial\xi^2}(\tfrac{1}{2}y^2 - \chi)\right]_{\xi=0,\,\alpha} \sim \tfrac{1}{2}\pi\,\mathrm{Re}\{\lambda_1 \exp(-\lambda_1 v)\},$$

and from (4.7), $$\left[(\cosh\xi - \cos\eta)\frac{\partial^2}{\partial\xi^2}\phi\right]_{\xi=0,\,\alpha} \sim \pm\frac{6}{\alpha^2}(1 - \cos\alpha v),$$

where $\eta = \alpha v$. On inserting the factor $M$ given by (4.12), it is clear that when

$$(15\pi)^{-1}\alpha^3(1 - \cos\alpha v) \ll |\lambda_1 \exp(-\lambda_1 v)|,$$

the flow separates from the cylinder and the plane and the separation points will be approximately those determined in § 3. The outermost separation point on the cylinder is that from which emanates the streamline which divides the fluid passing over the cylinder from that flowing through the gap between the cylinder and the plane. This streamline is asymptotic to $y = (2M)^{\frac{1}{2}}$ as $|x| \to \infty$. As $\alpha$ is decreased, the additional separation points appear in pairs since a separation streamline must begin and end on the same boundary.

To determine the location of the points of separation for general values of $\alpha$ and to establish at what distance the cylinder must be from the plane for separation to occur, it is necessary to determine the zeros of (4.14) and (4.15) and these have to be solved numerically. Because of the symmetry of the flow about the plane $x = 0$, we need consider only solutions with $0 \leqslant \eta \leqslant \pi$. Separation first occurs at the largest double zero of (4.14) or (4.15). The results of our numerical work show that separation first occurs from the plane when $\alpha = 1{\cdot}1123$, which corresponds to a value of $d/a \approx 1{\cdot}685$. The point on the plane where separation starts has a dimensionless $x$ co-ordinate of 1·148. It will be noticed that this differs little from 1·117, which is the value of $x$ at the outermost separation point on the plane when the cylinder is in contact with the plane. When $d/a$ is decreased below this critical value, a single eddy forms on the plane, the point of reattachment of its bounding streamline to the plane being such as to approach the point with co-ordinate 0·433, this being the second outermost separation point on the plane given in table 3. Separation from the cylinder begins when $\alpha = 0{\cdot}2448$, giving $d/a \approx 1{\cdot}030$, and the point on the cylinder where separation first occurs has Cartesian co-ordinates $x = 0{\cdot}341$, $y = 0{\cdot}149$. A single eddy forms and grows from this point as $\alpha$ is decreased. A second eddy forms on the plane when $\alpha = 0{\cdot}046$. With further decrease in the value of $\alpha$, the second eddy forms on the cylinder followed by a third eddy on the plane, and so on, with the eddies forming alternately on the plane and the cylinder. As the eddies grow, the dividing streamlines become closer together, but since the fluid which upstream lies between the planes $y = 0$ and $y = (2M)^{\frac{1}{2}}$ must pass through the gap between the cylinder and plane, it is forced to 'snake' its way

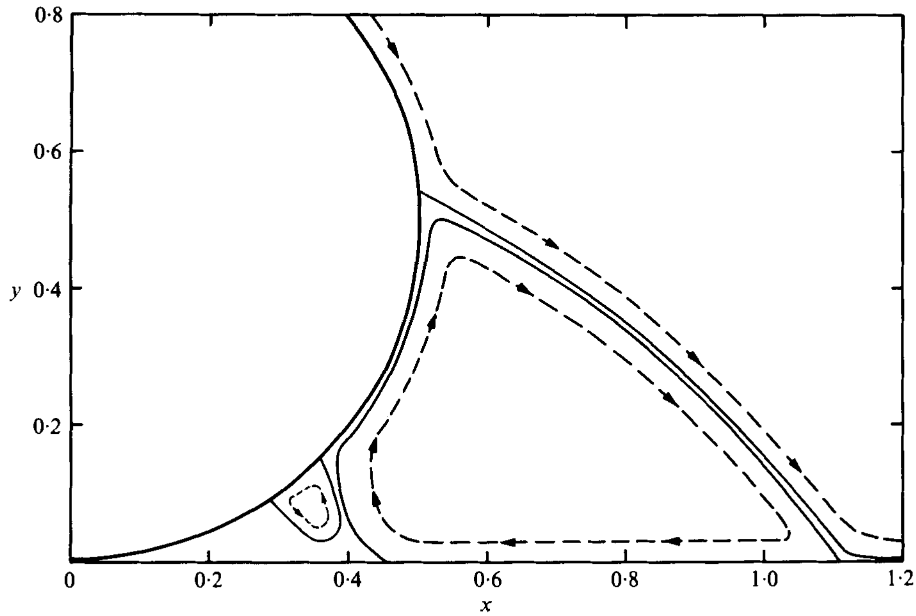


FIGURE 2. The dividing streamline and eddies when $\alpha = 0{\cdot}15$.

between the interlacing eddies which form on either side of the plane $x = 0$. In this way we can see how the infinite nest of eddies which exists in the flow when the cylinder actually touches the plane is formed in a systematic manner as the cylinder approaches contact with the plane.

It is somewhat surprising that although separation from the plane occurs when the minimum gap between the cylinder and the plane is $0{\cdot}685a$ or less, separation from the cylinder does not start until the gap is reduced to 3 % of the cylinder radius, and it is when this already small gap approaches zero that the infinite set of nested eddies is produced in the flow. In figure 2, we have drawn the separation streamlines for $\alpha = 0{\cdot}15$, for which $d/a \approx 1{\cdot}01$. At this value of $\alpha$, separation occurs on both the plane and the cylinder, but there is only one eddy on either boundary. Again the broken line indicates the general direction of the flow.

## 5. Force and torque acting on the cylinder

In a Stokes flow past a fixed cylinder of radius $a$ with velocity components given by (2.2), it may be shown that the components of force acting on the cylinder are given by

$$X = 2\mu U a \oint \left( -y \frac{\partial}{\partial n} \nabla^2 \psi + \frac{\partial y}{\partial n} \nabla^2 \psi \right) ds, \tag{5.1}$$

$$Y = 2\mu U a \oint \left( x \frac{\partial}{\partial n} \nabla^2 \psi - \frac{\partial x}{\partial n} \nabla^2 \psi \right) ds,$$

whilst the torque $T$ about the axis of the cylinder is

$$T = 2\mu U a^2 \oint \nabla^2 \psi \, ds, \tag{5.2}$$

where $T$ and the arc length $2as$ are measured clockwise and the normal is drawn into the fluid. In the cases considered in the previous sections, $\nabla^2\psi$ and $\partial/\partial n$ are even in $x$, so that $Y = 0$.

Noting that $y$ and $\nabla^2\psi$ are harmonic functions, the integral for $X$ can be taken round any closed path in the fluid encircling the cylinder once. Further, since $\partial y/\partial n = \partial x/\partial s$, it follows that

$$X = -2\mu Ua \oint \left[\left(y\frac{\partial}{\partial n} + x\frac{\partial}{\partial s}\right)\nabla^2\psi\right] ds. \tag{5.3}$$

This formula shows immediately that the contribution from the term $\frac{1}{2}y^2$ in $\psi$ to $X$ is identically zero.

An advantageous choice of contour for the remaining integral would appear to be the $x$ axis and a large semicircle at infinity but the order of magnitude specified for $\psi - \frac{1}{2}y^2$ at large distances does not immediately confirm that the contribution from the large semicircle to $X$ tends to zero. We shall take an arbitrary circle in the relevant co-ordinate system and obtain an expression for $X$ independent of the circle chosen.

When the cylinder is not in contact with the plane, we have, from (4.1). (4.2), (4.6) (4.7) and (5.3),

$$\begin{aligned}\frac{X}{2\mu Ua} &= \int_{-\pi}^{\pi}\left\{\left(y\frac{\partial}{\partial \xi} - x\frac{\partial}{\partial \eta}\right)\nabla^2[(\cosh\xi - \cos\eta)^{-1}f]\right\} d\eta \quad (0 < \xi \leqslant \alpha) \\ &= \int_{-\pi}^{\pi}\left\{\left(\sinh\xi\frac{\partial}{\partial \xi} - \sin\eta\frac{\partial}{\partial \eta}\right)\nabla^2[(\cosh\xi - \cos\eta)^{-1}f]\right\}\frac{c\,d\eta}{\cosh\xi - \cos\eta},\end{aligned} \tag{5.4}$$

where

$$\begin{aligned} f(\xi,\eta) &= -\tfrac{1}{2}c^2\sum_{n=0}^{\infty}\chi_n(\xi)\cos\eta + M[\phi_0(\xi) + \phi_1(\xi)\cos\eta] \\ &= \sum_{n=0}^{\infty} f_n(\xi)\cos n\eta. \end{aligned} \tag{5.5}$$

Equation (5.4) can be simplified to

$$\begin{aligned}\frac{X}{2\mu Ua} &= \frac{1}{c}\int_{-\pi}^{\pi}\left[(\cosh\xi + \cos\eta)\left(\frac{\partial^2 f}{\partial \eta^2} - \frac{\partial^2 f}{\partial \xi^2} + f\right) - \left(\sinh\xi\frac{\partial f}{\partial \xi} + \sin\eta\frac{\partial f}{\partial \eta}\right)\right. \\ &\qquad \left. + \left(\sinh\xi\frac{\partial}{\partial \xi} - \sin\eta\frac{\partial}{\partial \eta}\right)\left(\frac{\partial^2 f}{\partial \xi^2} + \frac{\partial^2 f}{\partial \eta^2}\right)\right] d\eta \\ &= (2\pi/c)\{\sinh\xi\,[f_0'''(\xi) - f_0'(\xi)] - \cosh\xi\,[f_0''(\xi) - f_0(\xi)]\}.\end{aligned}$$

Further use of (5.5) and substitution of (4.8) for $\chi_0$ and $\phi_0$ yields

$$X/2\mu Ua = (4\pi/c)(\tfrac{1}{2}c^2A_0 - Ma_0).$$

Then since $c = \frac{1}{2}\sinh\alpha$ and $M$ is given by (4.11), we obtain

$$X = 4\pi\mu Ua(\sinh\alpha/\alpha), \tag{5.6}$$

which in terms of the gap width $\epsilon = \frac{1}{2}(\cosh\alpha - 1)$ may be written

$$X = 4\pi\mu Ua(1 + \tfrac{2}{3}\epsilon - \tfrac{4}{45}\epsilon^2 + \dots).$$

The contribution of the $\frac{1}{2}y^2$ term to the clockwise moment $T$, given by (5.2), is clearly $2\pi\mu Ua^2$. Accordingly, $T$ is given by

$$\frac{T}{2\pi\mu Ua^2} - 1 = \frac{1}{\pi}\int_{-\pi}^{\pi} \{\nabla^2[(\cosh\xi - \cos\eta)^{-1}f]\}_{\xi=\alpha} \frac{c\,d\eta}{\cosh\alpha - \cos\eta}$$

$$= \frac{1}{\pi c}\int_{-\pi}^{\pi} \left\{\frac{\partial^2 f}{\partial\xi^2} + \frac{\partial^2 f}{\partial\eta^2} - f + \frac{2f\cosh\alpha}{\cosh\alpha - \cos\eta} - \frac{2}{\cosh\alpha - \cos\eta}\left(\sinh\alpha\frac{\partial f}{\partial\xi} + \sin\eta\frac{\partial f}{\partial\eta}\right)\right\}_{\xi=\alpha} d\eta$$

$$= \frac{2}{c}\{f_0''(\alpha) - f_0(\alpha)\} + \frac{2}{c^2}\cosh\alpha \sum_{n=0}^{\infty} f_n(\alpha)\,e^{-n\alpha} - \frac{2}{c^2}\sinh\alpha \sum_{n=0}^{\infty}[f_n'(\alpha) - nf_n(\alpha)]\,e^{-n\alpha}.$$

The formulae for $\chi_n(\xi)$ $(n \geqslant 0)$, $\phi_0(\xi)$ and $\phi_1(\xi)$ imply that

$$f_0''(\alpha) - f_0(\alpha) = \tfrac{1}{4}\sinh\alpha = \tfrac{1}{2}c$$

whilst the remaining terms above yield

$$-4\sinh^2\alpha \sum_{n=1}^{\infty} n\,e^{-2n\alpha} = -1.$$

Hence the contribution to the torque from the disturbance flow due to the presence of the cylinder is zero and

$$T = 2\pi\mu Ua^2, \tag{5.7}$$

which indicates that $T$ is independent of $\alpha$ and therefore of the gap width $\epsilon$.

Similarly it may be shown that in the contact case of §3,

$$X = 4\pi\mu Ua, \quad T = 2\pi\mu Ua^2,$$

which results are consistent with (5.6) and (5.7) and agree with those of Schubert (1967).

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